Commutative closures of regular languages

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Outline

1. **Trace monoids and recognizability**
   - The free partially commutative monoid
   - Recognizability in non-free monoids
2. **Varieties**
   - Relationship with recognizable languages
3. **Commutative closures**
   - Closure under P_4
4. **References**

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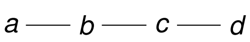

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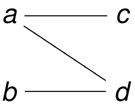

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- If $t \in \mathbb{M}$, a word w such that $t = [w]_I$ is a linearization of t.

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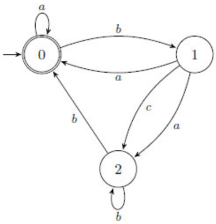

$$(a^*(ba)^*(b(a+c)b^*b)^*)^*$$

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Recognizable $\neq$ rational

$(\Sigma, I) = a$—b, $L = (ab)^*$.
$\phi^{-1}(L) = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$

Syntactic monoid

Let M be a monoid and $T \subseteq M$. The syntactic preorder over T, $\leq_T$ is defined by:

$$x \leq_T y \text{ if } \forall u, v \in M, uyv \in T \Rightarrow uxv \in T$$

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Characterization of recognizable languages

A subset $T \subseteq M$ is recognizable if and only if, its syntactic monoid M_T is finite.

Varieties of monoids and recognizable languages

Variety of monoids

A class of monoids closed under taking submonoids, left and right quotients and direct products.
A *pseudovariety of monoids* is a variety of finite monoids.

Example

The class of all finite monoids is a pseudovariety.
The class of finite groups is a pseudovariety denoted **G**.

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Variety theorem

The correspondance $\mathbf{V} \to \mathcal{V}$ that maps to a variety of monoids $\mathbf{V}$ the variety of languages whose syntactic monoid is in $\mathbf{V}$ is one to one.

Polynomial closure

The polynomial closure $Pol(L)$ of a class of language $\mathcal{L}$ over Σ is the set finite unions of $L_0 a_1 L_1 a_2 \cdots a_n L_n$ whith $a_i \in \Sigma$ and $L_i \in \mathcal{L}$.

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Polynomials of group languages

- $Pol(\mathcal{G})$ (the polynomial closure of the variety associated to **G**) is closed unter total commutation.
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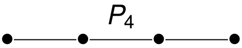

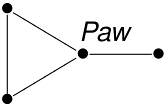

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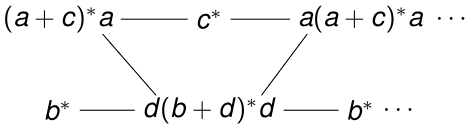

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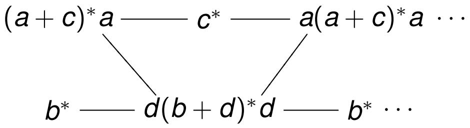

Proof

- Proof relies on structure of traces.
- Recognizability is shown using MSO logic
- See the internship report for more details.

References

Gómez, Antonio Cano and Guaiana, Giovanna and Pin, Jean-Éric
When Does Partial Commutative Closure Preserve Regularity?
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